Probabilistic Vibration Analysis of Nearly Periodic Structures

Karen S. Hamade* and Efstratios Nikolaidis†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

A disordered structure with weakly coupled subsystems has localized modes with the vibratory energy concentrated in one part of the mode shape. The localization of modes may significantly affect the forced vibration response, increasing the maximum vibration amplitude dramatically in some cases. It is the scope of this paper to study the forced vibration of nearly periodic systems consisting of almost identical substructures and to analyze their free and forced dynamic responses probabilistically. It is shown that the sensitivity of the forced response to the degree of localization depends on a combination of the symmetry of the mode, which is excited, and the phase difference between the forces acting on each substructure. These results might explain the range of contrasting conclusions of earlier publications on the effects of forced response on mistuned structures. A probabilistic analysis of the free and forced responses of a nearly periodic structure is shown to be useful in the design of structures that are sensitive to the degree of localization. The results of the probabilistic analysis are verified using Monte Carlo simulation.

Nomenclature

A	= localization factor
$a_i(t)$	= generalized coordinate
$\bar{a}_i$	= ith modal amplitude
a	= larger localized modal amplitude
a^*	= smaller localized modal amplitude
c	= stiffness of beam's torsional spring
$\bar{c}$	= nondimensionalized spring stiffness
$\bar{c}'$	= nondimensionalized stiffness in reduced space
EI	= beam flexural rigidity
$F_{1,2}$	= applied harmonic forces
$g(X)$	= performance function
k	= stiffness of linear spring coupling pendula
L	= beam length
$L_{1,2}$	= pendula support lengths
m	= mass of single-span beam
NM	= number of tuned modes in Rayleigh-Ritz method
P_F	= probability of failure
P_L	= probability of localization
P_s	= probability of survival
W_{nc}	= nonconservative work of the applied forces
$w(x,t)$	= beam vertical displacement
$\bar{w}(x,t)$	= nondimensional beam displacement
X	= array of design variables
$\bar{x}$	= nondimensional position along beam length
$x_{1,2}$	= pendula displacements
x_c	= position of beam's center support
$Y(x_1, x_2, \omega)$	= transfer admittance
α	= phase difference between applied forces
$\beta_{1,2}$	= Lagrange multipliers
$\bar{\beta}_{1,2}$	= harmonic amplitude of Lagrange multipliers
$\Delta_{1,2}$	= damping factor between substructures
ΔL	= length difference between beam spans, or mistuning
$\overline{\Delta L}$	= nondimensional mistuning
$\overline{\Delta L}'$	= nondimensional mistuning in reduced space
$\mu_{\bar{c}}$	= mean of nondimensional spring stiffness of beam
$\mu_{\overline{\Delta L}}$	= mean of nondimensional mistuning
$\phi_i(x)$	= modes of the single-span beam
$\Psi_i(x)$	= modes of the mistuned beam
$\sigma_{\bar{c}}$	= mean of nondimensional spring stiffness of beam
$\sigma_{\overline{\Delta L}}$	= mean of nondimensional mistuning
ω_i	= frequencies of the single-span beam
$\omega_{1,2}$	= tuned frequencies of coupled pendula
Ω	= frequency of forced excitation of beam and pendula
$\bar{\Omega}$	= nondimensional forcing frequency
$\Omega_{1,2}$	= modal frequencies of disordered beam and pendula systems
$\bar{\Omega}_{1,2}$	= nondimensional modal frequencies

I. Introduction

A PERIODIC structure consists of a number of identical substructures and exhibits cyclic symmetry or some other form of periodicity. A nearly periodic structure is one that has a slightly altered property between its substructures. An example is a two-span beam with the two span lengths differing by a small fraction of the beam length (Fig. 1). Another example is a system of coupled pendula with slightly varied supporting lengths (Fig. 2). The disorder in these systems is called structural mistuning. In general, mistuning is the small variation in the dynamic properties of a system. It is usually caused by manufacturing imperfections or general degradation due to aging. When disorder is introduced into a structure, the result is a concentration of vibrational energy in one part of the mode shapes. This phenomenon, called localization of modes, may occur in any mode shape.

Localization of modes is important because the dynamic response of a mistuned system may be considerably higher than that of a tuned system, leading to higher vibration levels and larger stresses. It has been found that the small differences in the structural or inertial properties of the structure can affect the amplitudes of individual substructures by several hundred percent, which can result in structural failure[1,2] Since localization occurs for even small deviations of periodicity in

Received April 15, 1988; revision received Feb. 2, 1989.

*Graduate Student, Department of Aerospace and Ocean Engineering; currently, Aerospace Engineer, NASA Ames Research Center, Moffett Field, CA. Member AIAA.

†Assistant Professor, Department of Aerospace and Ocean Engineering.

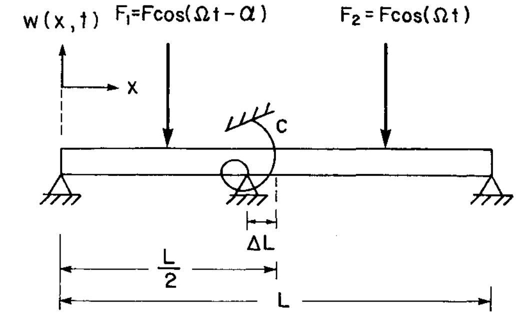

Fig. 1 Forced excitation of the disordered two-span beam.

the structure, prediction of the response of a mistuned structure is particularly important. In fact, mistuning played an important role in several costly failures in the development and production of modern aircraft turbofan engines.[3]

The coupling between the substructures of a nearly periodic structure is an important parameter. When the substructures are weakly coupled, localization is more likely to occur.[4–6] On the other hand, if the coupling between subsystems is too strong, the mode shapes will not be localized, even for high degrees of mistuning. In the case of the two-span beam (Fig. 1), the two segments of the beam are coupled since the beam is continuous across its length. A torsional spring at the position of the middle support is adjusted to change to the amount of coupling in the system. Similarly, coupling in the case of the two pendula in Fig. 2 is due to the linear spring that connects them.

One of the earliest studies of the localization of mistuned bladed disk assemblies was done in 1966 by Whitehead.[1] Since then, the free vibration of nearly periodic structures has been studied extensively.[4,7,8] For example, Pierre[4] investigated the free vibration in a two-span beam and experimentally verified the existence of localized modes. Less work has been done in the area of the forced response of mistuned systems. Sinha[9] and Huang[10] estimated the statistics of the forced response of mistuned bladed disk assemblies. However, the accuracy and applicability of Sinha's method is limited by the assumption that the amount of damping and mistuning in the system is small. Furthermore, Huang focused on the first two moments of the forced response without considering other important statistical quantities, such as the probability that the maximum vibratory amplitude of a disk assembly exceeds some critical value. Monte Carlo simulation has also been used to estimate the forced response statistics of mistuned bladed disk assemblies. Yet it has not been possible to obtain large samples necessary for reliable estimates of the response statistics due to the excessive computational effort required in such simulation studies.[2]

Bendiksen[6,11] studied the localization phenomenon in large space structures such as astronomical telescope reflectors and communication antennas. However, his work concentrated on free vibration.

A limited amount of work toward the understanding of the localization phenomenon has been done using simpler nearly periodic structures. Hodges[5] illustrated the phenomenon of localization in nearly periodic structures using a vibrating string with irregularly spaced masses. A study of the propagation of vibration through the structure showed that there is a confinement of energy close to the source of excitation with an exponential decay away from the driving point. These results were verified by Pierre using a system of coupled pendula.[12]

Conflicting results have been found in the area of the forced response of nearly periodic structures. Most research has shown that an increase in the degree of mistuning in a bladed disk assembly causes an increase in the response amplitude[1,2,13,14]; however, there have also been conclusions drawn to the contrary.[15,16] For example, Sogliero and Srinivasan[17] found that an increase in mistuning caused a decrease in the forced response, hence an increase in the fatigue life of the blades.

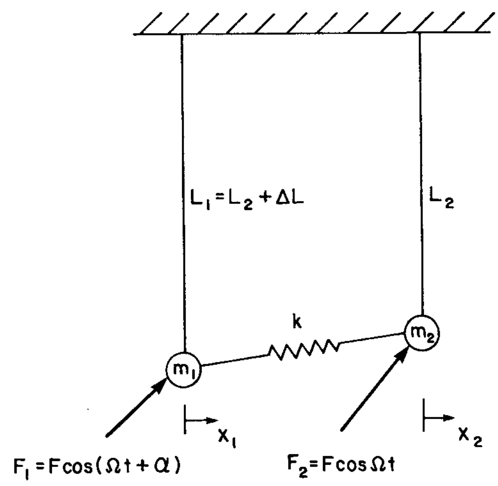

Fig. 2 Forced excitation of disordered coupled pendula.

There is currently a lack of understanding of the localization phenomenon in mechanical systems. Furthermore, it is widely accepted that a complete approach to the problem of mistuning should be probabilistic.[2] The scope of such an approach would be to derive the statistics of the vibratory stress and displacements from the tolerances in the dynamic parameters of the individual subsystems. The purpose of this work is to develop an approach for estimating the statistics of the free and forced vibration responses as well as the probability of failure due to excess vibration levels of some simple disordered structural systems. Furthermore, attention is given to a loading case that has been ignored by previous studies: that of a harmonic load acting simultaneously in phase or out of phase on each substructure. This is an important loading case for two reasons. First, it simulates the loading applied to some real-life engineering structures such as bladed disk assemblies. Second, the effects of mistuning on the forced response level are different from those in the loading cases considered by previous studies. Consideration of the mentioned loading cases may provide an explanation of the conflicting conclusions found in these studies concerning the effect of mistuning on the dynamic response.

The objectives of this paper are as follows:

1) To study the statistical characteristics of the localized modes of a nearly periodic structure (the two-span beam in Fig. 1) for random configurations of mistuning and coupling in the system.

2) To obtain the dynamic response in the case of two in- and out-of-phase loads acting simultaneously at different parts of the structure.

3) To calculate the probability of failure for a range of failure limits given that the system is randomly configured in its degree of mistuning and coupling of the substructures.

4) To study the sensitivity of the probability of failure due to excessive vibratory levels to the statistics of the tolerances in some of the system parameters.

First, the forced vibration response is studied. Both in-phase and 180-deg out-of-phase forces that excite the first or second mistuned mode shape are considered. It is shown that the forced response of a localized structure may be sensitive or relatively insensitive to the degree of localization, depending on the mode being excited and the phase difference between the applied forces. This last observation might explain the conflicting conclusions of previous studies concerning the effect of mistuning on the dynamic response of bladed disk assemblies. Next, a probabilistic analysis of the free and forced vibration is performed using the second-moment method[18] and assuming random amounts of coupling and mistuning in the system. The probability of high localization of the modes of the two-span beam is evaluated. The results are verified using Monte Carlo simulation. In the forced vibration analysis, the probability of failure is calculated due to excess vibration levels.

II. Forced Response of Nearly Periodic Structures

The understanding of the forced response of nearly periodic structures is far from being complete. Nevertheless, the forced response prediction is crucial in evaluating the reliability and expected life of such structures. In the present study, the excitation loads are assumed to be deterministic. Randomness is introduced into the problem only through the random deviations of coupling and mistuning in the two-span beam and the pair of coupled pendula. Two applied forces having equal amplitudes and a phase difference α are applied at two points. These are given by the equations

$$F_1 = F\cos(\Omega t - \alpha) \quad \text{and} \quad F_2 = F\cos(\Omega t) \tag{1}$$

where the phase difference α is either 0 or 180 deg.

Figure 1 shows the two-span beam system composed of a simply supported beam of length L with Young's modulus E, mass per unit length m, and moment of inertia I. The torsional spring is located at x_c, the position of the central support. This support can be moved to the left or right of center of the beam to introduce a disorder ΔL, which has a nondimensional value $\overline{\Delta L} = \Delta L/L$. This disorder has a range of values of 0–7% of the beam length. In addition, a nondimensional spring stiffness $\bar{c}$ is defined such that $\bar{c} = 2cL/EI$, which varies from 200 to 1000. The first and second modes can be calculated as in Ref. 4 by a Rayleigh-Ritz technique. In this study the deflection of the two-span beam was expressed as a weighted sum of the known modes NM of a single-span beam of length L as follows:

$$w(x, t) = \sum_{i=1}^{NM} a_i(t)\phi_i(x) \tag{2}$$

where the terms $a_i(t)$ are the generalized coordinates of the free vibration problem. The modes of the single-span beam are

$$\phi_i(x) = \sqrt{\frac{2}{mL}}\sin\left(\frac{i\pi x}{L}\right) \tag{3}$$

with corresponding natural frequencies $\omega_i = (i\pi)^2\sqrt{EI/mL^4}$.

This analysis is extended here to the forced response where the forces are located at one- and three-quarters of the beam length (Fig. 1). The frequency of the applied forces, Ω, is set close to the frequency of the mode to be excited, Ω_i such that

$$\bar{\Omega} = \bar{\Omega}_i - 0.0001, \qquad i = 1, 2 \tag{4}$$

where $\Omega = \bar{\Omega}\sqrt{EI/mL^4}$. The forced response of the two-span beam is calculated by the Rayleigh-Ritz method as described in the Appendix. The results are also verified using the mode superposition method. This method finds the beam displacement

$$w(x, t) = \sum_{i=1}^{N} \Psi_i(x)q_i(t) \tag{5}$$

where the Ψ_i are the mistuned modes of the beam, the q_i are generalized coordinates, and N is the number of localized modes used in the summation. Since the frequency of the applied force is approximately equal to the frequency of one mode, only the contribution to the response from that mode is considered in Eq. (5). An analysis is done to study the changes in the response of the disordered two-span beam when the degree of mistuning and coupling is changed in the system. A large spring stiffness corresponds to low coupling between the two spans. Furthermore, the degree of mistuning increases with ΔL. A combination of low coupling and a high degree of mistuning increases the possiblity of having strongly localized modes.

First an analysis is made of the response of the disordered two-span beam due to in- and out-of-phase forces exciting the

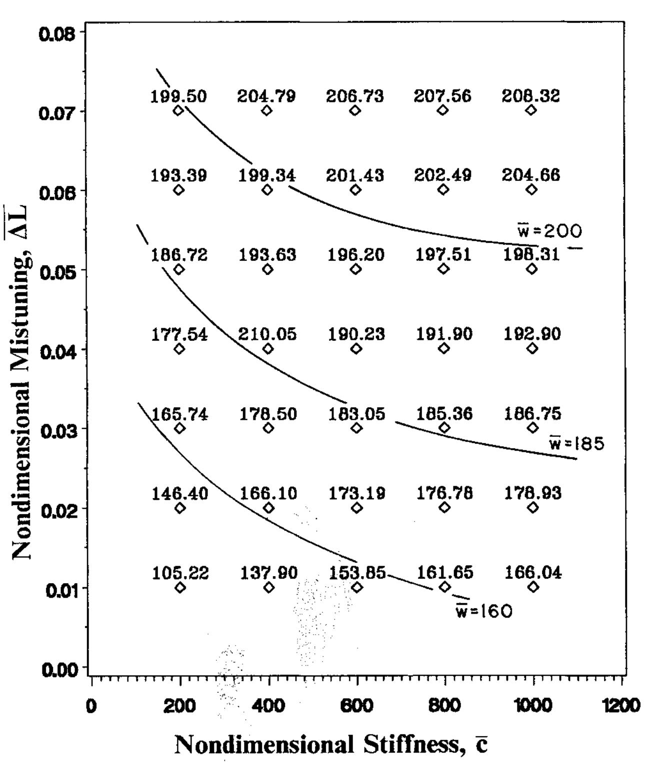

Fig. 3 In-phase forced response of the beam exciting the first mode.

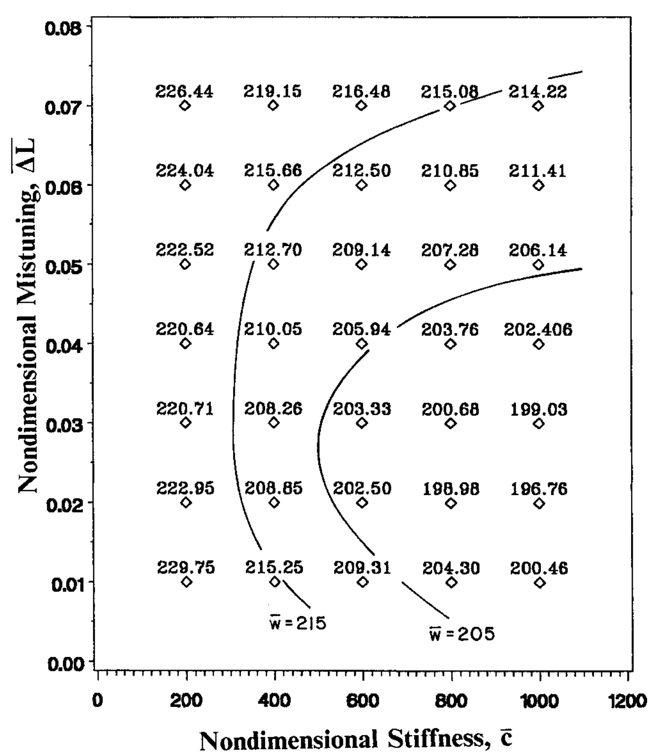

Fig. 4 Out-of-phase forced response of the beam exciting the first mode.

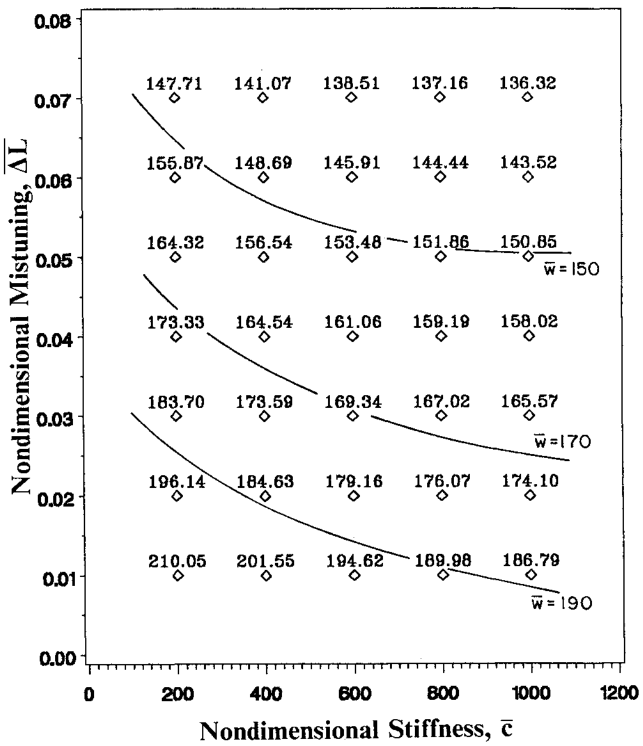

Fig. 5 In-phase forced response of the beam exciting the second mode.

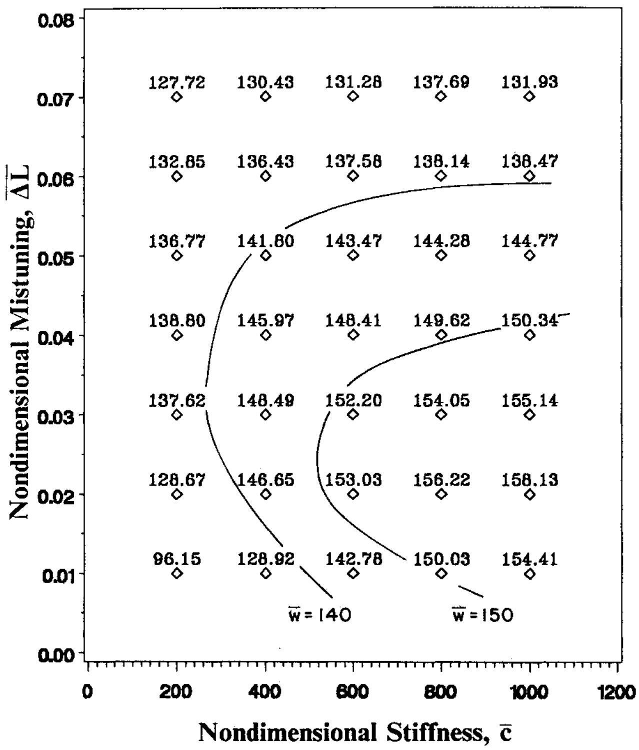

Fig. 6 Out-of-phase forced response of the beam exciting the second mode.

first mode. Figure 3 shows the results for in-phase forces. Here, the values of the maximum amplitude of the nondimensional response, $\bar{w}$, is plotted for varying amounts of coupling and mistuning in the structure. The curves are the loci of points with constant maximum response as calculated in the Appendix. The important conclusion drawn from this figure is that the maximum response increases monotonically with spring stiffness $\bar{c}$ and degree of disorder $\overline{\Delta L}$. Therefore, the in-phase forced excitation of the first mode results in a response that increases with the degree of localization in that mode. Next, the case of 180-deg out-of-phase forces exciting the first mode is considered (Fig. 4). In this system, an increase in the localization of the first mode of the beam results in a slight decrease in the maximum response and is relatively insensitive to the degree of localization. This is opposite to the previous case of in-phase forces already discussed.

A similar comparison is made of responses due to in- and out-of-phase forces exciting the second mode of the two-span beam. For in-phase forces, the maximum response decreases and is relatively insensitive as the mistuning and coupling between substructures increases (Fig. 5). On the other hand, the maximum response due to two 180-deg out-of-phase forces exciting the second mode increases with localization and is sensitive to these changes (Fig. 6). These conclusions are opposite to those reached in the case where the first mode is excited.

Similar conclusions are drawn by considering a pair of coupled pendula such as that in Fig. 2. This system consists of two nearly identical substructures coupled by a linear spring. As in the two-span beam case, the response of the system is either sensitive or relatively insensitive to the degree of localization of the excited mode, depending on the symmetry of the mode and the phase difference of the applied forces. The beam and pendula systems are compared in terms of mode symmetry because in the first mode, which is symmetric, the pendula move in tandem, a case comparable to that of the symmetric second mode of the two-span beam. Similarly, both the second mode of the pendula and the first mode of the two-span beam are antisymmetric. Considering the in-phase excitation of the first mode, it can be shown that an increase in the localization of the first mode causes the forced response due to in-phase forces to increase; however, these changes are relatively insensitive to the degree of localization in that mode. On the other hand, 180-deg out-of-phase forces cause the pendula displacements to increase drastically with an increase in the degree of localization. Thus, the effect of the degree of localization on the response is opposite to that in the preceding case.

It is hypothesized that previous studies that showed a large forced response for highly localized systems used a model with a combination of forcing phases and excited modes that results in this sensitivity. In contrast, studies showing a response that is relatively insensitive to localization may have been based on a case such as that of the in-phase excitation of an antisymmetric mode.

A. Comparison of Forced Response Analyses

In brief, the analysis indicates that if the modes are strongly localized, the in-phase excitation of an antisymmetric mode and the out-of-phase excitation of a symmetric mode will result in a large forced response. However, the in-phase excitation of a symmetric mode and the out-of-phase excitation of an antisymmetric mode result in small changes in the forced response.

The transfer admittance of a disordered structure is useful in the analysis of the forced response because it indicates the amount of vibratory energy that passes between two nearly periodic substructures. It is used here to explain the results of the forced response analysis of the coupled pendula. In general, the transfer admittance $Y(x_1, x_2; \omega)$ is defined as the complex vibratory amplitude at x_1 due to a unit amplitude harmonic force of frequency ω applied to x_2. It indicates the lack of resistance with which energy passes within a structure. For the case of two coupled pendula, the transfer admittance is[5]

$$Y(x_1, x_2; \omega) = \sum_{r=1,2} \frac{j\omega \Psi_r(x_1)\Psi_r(x_2)}{\Omega_r^2 + 2j\omega\,\Delta_r - \omega^2} \tag{6}$$

where x_1 and x_2 are the displacements of the first and second masses, respectively, and j is the square root of -1. The rth mode is $\Psi_r(x)$ with an associated natural frequency Ω_r. The term ω is the frequency of excitation, Δ_r is the damping factor, and NM is the number of modes used in the summation. For two in-phase forces F_1 and F_2, the displacements of the first and second masses can be expressed as

$$x_1 = Y_{1,1}F_1 + Y_{1,2}F_2 \tag{7}$$

$$x_2 = Y_{2,1}F_1 + Y_{2,2}F_2 \tag{8}$$

where Y_{ij}, $i, j = 1, 2$ is the transfer admittance between x_i and x_j. For a strongly localized system, the modal amplitude of one pendulum is small so that the product of mode amplitudes in Eq. (6) is almost zero. As a result, the terms $Y_{1,2}$ and $Y_{2,1}$ become small and have less contribution to the transfer of energy between m_1 and m_2. However, the transfer admittance terms $Y_{1,1}$ and $Y_{2,2}$ do not show this large decrease.

For the case where the forces are applied to the pendula in phase with a frequency equal to the first natural frequency, Ω_1, the term in the sum of Eq. (6) corresponding to the first mode dominates. Since the masses in the first mode move in tandem, $\Psi_1(x_1)$ and $\Psi_1(x_2)$ have the same sign, and the forces F_1 and F_2 are acting in phase with equal amplitudes. Then the terms $Y_{1,2}F_2$ and $Y_{2,1}F_1$ in Eqs. (7) and (8) contribute to increasing displacement, and the responses of the pendula are relatively insensitive to the degree of localization. This explains the results obtained from the forced vibration solution of the beam shown in Fig. 5. For the case of 180-deg out-of-phase forces exciting the first mode, the two terms in Eqs. (7) and (8) have opposite signs. Therefore, the contribution of one force to the displacements opposes the other. If the systems were ordered, the out-of-phase forces would cancel the energy of an asymmetric mode. However, the introduction of disorder limits the cancellation because the transfer admittance is low. Therefore, localization eliminates some of the cancellation of energy in a way such that the response is significantly higher than that of a tuned system. This trend of forced response is in agreement with that observed in Fig. 3 for the two-span beam.

III. Probabilistic Analysis of Nearly Periodic Structures

It is important to calculate the statistics of the dynamic response of a system because in practice the structural parameters of nearly periodic systems vary in a random fashion. With each possible configuration of the system, there is a different degree of localization and maximum response amplitude. Also, the vibration of nearly periodic structures can be very sensitive to the amount of mistuning; therefore, small deviations in the system parameters from their nominal values could drastically affect the dynamics of the system. In this analysis we assume that the variables expressing the degree of disorder and coupling are statistically independent Gaussian random variables with known first and second moments. A second-moment method is used in this analysis. Although a fully probabilistic method could have been used, we deemed that a second-moment method should be employed since it has the ability to reduce the computational effort dramatically. Therefore, this latter approach is more appropriate than fully probabilistic methods for real-life structures consisting of a large number of substructures.

A. Second-Moment Method

The method of measuring reliability as a function of the first and second moments is the second-moment method,[18] which was formulated by Cornell[19] in 1969 and later in 1974 by Ang and Cornell[20] and Hasofer and Lind.[21] In this method the performance function $g(X) = g(x_1, x_2, \ldots, x_n)$ defines the regions of failure and survival where X is the vector of n design variables, $X = x_1, x_2, \ldots, x_n$. In this generalized form, $g(X) > 0$ defines a safe system, $g(X) < 0$ defines an unsafe system, and $g(X) = 0$ is the limit-state equation. In the example of the two-span beam, the vector of design variables X is composed of the spring stiffness $\bar{c}$ and mistuning $\overline{\Delta L}$. These are assumed to be normally distributed random variables.

The performance function can be expressed here in terms of the reduced design variables $\bar{c}'$ and $\overline{\Delta L}'$, which are standard Gaussian random variables. In terms of these variables, the limit-state equation is

$$g(c, \Delta L) = g(\sigma_{\bar{c}}\bar{c}' + \mu_{\bar{c}}, \sigma_{\overline{\Delta l}}\overline{\Delta L} + \mu_{\overline{\Delta L}}) = 0 \tag{9}$$

Equation (9) defines the minimum conditions for a localized mode in the space of the reduced design variables. Shinozuka[22] was the first to show that the most probable point of failure, $(\bar{c}'^*, \overline{\Delta L}'^*)$, lies on the failure surface and that this point has a minimum distance from the origin (Fig. 7.) This minimum distance is called the safety index, β. The safety index is given by Ref. 18:

$$\beta = -\frac{1}{D}\left[\bar{c}'^*\left(\frac{\partial g}{\partial c'}\right)_* + \overline{\Delta L}'^*\left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*\right] \tag{10}$$

where

$$D = \sqrt{\left(\frac{\partial g}{\partial c'}\right)_*^2 + \left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_*^2}$$

and the derivatives are calculated at the most probable failure point, $(\bar{c}'^*, \overline{\Delta L}'^*)$. The point $(\bar{c}'^*, \overline{\Delta L}'^*)$ can be expressed in terms of the safety index as $\bar{c}'^* = -\alpha_{\bar{c}}^*\beta$ and $\overline{\Delta L}'^* = -\alpha_{\overline{\Delta L}}^*\beta$. The terms α_c^* and $\alpha_{\overline{\Delta L}}^*$ denote the direction cosines along the $\bar{c}'$ and $\overline{\Delta L}'$ axes and are given by

$$\alpha_c^* = \frac{1}{D}\left(\frac{\partial g}{\partial \bar{c}'}\right)_* \quad \text{and} \quad \alpha_c^* = \frac{1}{D}\left(\frac{\partial g}{\partial \overline{\Delta L}'}\right)_* \tag{11}$$

These coefficients, also called sensitivity coefficients, indicate the importance of the associated random variable in evaluating the failure probability. For example, a value of the coefficient corresponding to a length variation close to 1 indicates that the failure probability increases rapidly with tolerance in length.

In this study the iterative procedure proposed by Rackwitz[23] is employed. The probability of localization P_L or failure P_F is given by the following expressions:

$$P_L = 1 - \Phi(\beta) \quad \text{and} \quad P_F = 1 - \Phi(\beta) \tag{12}$$

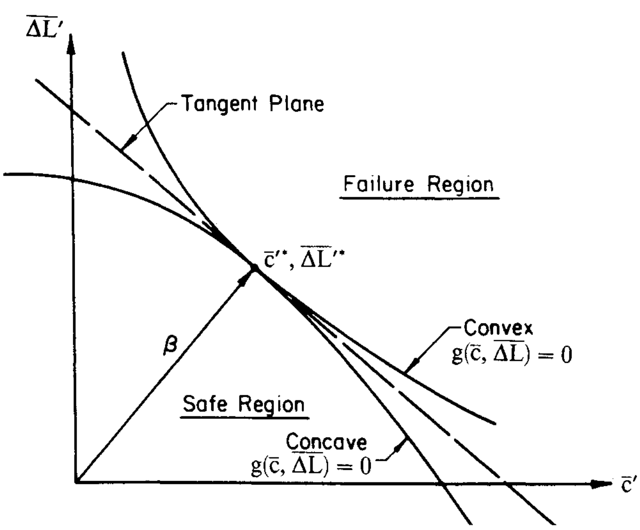

Fig. 7 Linear approximation of failure surface in reduced space.

where Φ is the probability distribution function of a standard normal random variable.[18]

B. Monte Carlo Simulation

The results of the probability of localization of the disordered two-span beam calculated by the second-moment method are checked using Monte Carlo simulation.[18] This method generates a set of random values of mistuning $\overline{\Delta L}$ and torsional stiffness $\bar{c}$ with the same mean and standard deviations used in the second-moment method. Each pair of randomly generated parameters is then used to calculate the localized mode shape A. Then, the number of localized states in the population divided by the total number of randomly generated points is equal to the fraction of localized states or the probability of localization P_L. In all cases very good agreement is found between the Monte Carlo simulation results and those obtained using the second moment method.

C. Probability of Localization

The second-moment method is used to calculate the probability of localization of the first mode of the two-span beam. The beam is assumed to have random amounts of coupling and mistuning. The mean value of stiffness is taken as $\mu_{\bar{c}} = 400$ with a standard deviation of $\sigma_{\bar{c}} = 40$. For the mistuning parameter, the mean value was taken as $\mu_{\overline{\Delta L}} = 0$ with a standard deviation of $\sigma_{\overline{\Delta L}} = 0.015$. The first tuned mode $(\overline{\Delta L} = 0)$ and the first localized mode with $\bar{c} = 400$ and $\overline{\Delta L} = 0.04$ are depicted in Fig. 8. Localization is arbitrarily defined as the event where the smaller amplitude a^* is below a given percentage of the larger amplitude a. This definition also defines the limit-state function. For example, if localization is defined to occur when the ratio of the lower to the higher amplitude of the mode shape is less than 0.1, meaning a localization factor $A = a^*/a \leq 0.1$, then the limit-state equation is $g(\bar{c}, \overline{\Delta L}) = A - 0.1 = 0$. The system is localized if $g \leq 0$. The middle support of the two-span beam can be moved to the left or right of center, and the localization factors are the same for positive or negative $\overline{\Delta L}$. As a result, there were two surfaces that defined localization. Therefore, the probability of the localization of Eq. (12) should be doubled in order to take into account two regions of localization to give

$$P_L = 2[1 - \Phi(\beta)] \tag{13}$$

Using the second-moment method, the distance from the origin to the surface defining localization, $A = 0.1$, is the safety index, $\beta = 1.502$. The corresponding probability of localization is 0.1336. The results for a range of localization factors of $A = 0.0$–0.3 are given in Fig. 9. This figure shows that the probability of localization increases for a less strict criterion of localization. For example, in the case of localization defined by a value of $A \leq 0.2$, the probability of localization of the first mode of the disordered two-span beam is 0.4658. However, when localization is set for $A \leq 0.1$, the probability of localization is 0.1336. The probability of localization tends to zero in the limiting case of complete concentration of energy in one-half of the mode shape ($A = 0.0$). The usefulness of this diagram is the ability to predict the probability that the modes of a system will be localized to a critical degree. It was observed that the results of the Monte Carlo simulation compare well with those of the second-moment method. For example, when localization was defined as $A \leq 0.1$, the Monte Carlo simulation using 700 simulations yielded a probability of localization of the first mode of the two-span beam equal to 0.1300.

D. Probabilistic Forced Vibration Analysis

In the previous section, the free vibration response of the two-span beam was studied and the probability of localization of the first mode was calculated. Now the forced excitation of the beam's first mode with two simultaneously applied in-phase forces is studied in terms of the probability of failure. This is a case where the response levels showed a drastic increase with an increase in localization (Fig. 3). An analysis of the probability of failure is important because a consider-

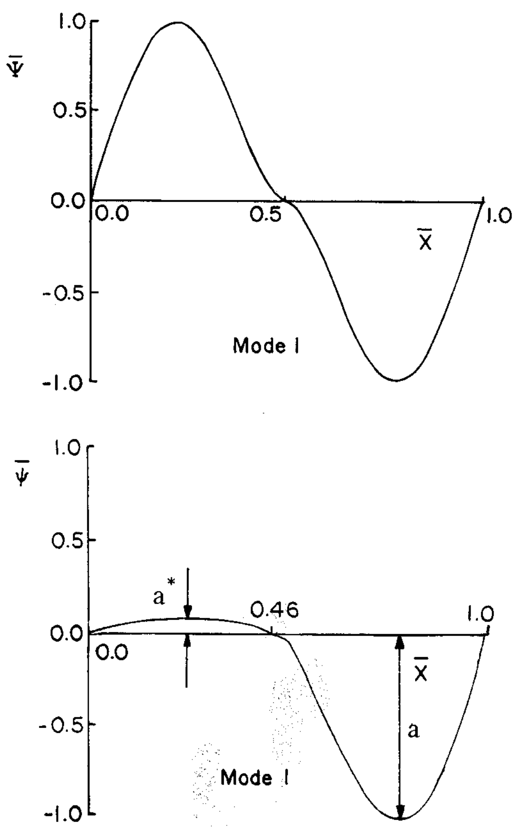

Fig. 8 First tuned and mistuned beam of the two-span beam; $\bar{c} = 400$, $\overline{\Delta L} = 0.04$ of the localized system.

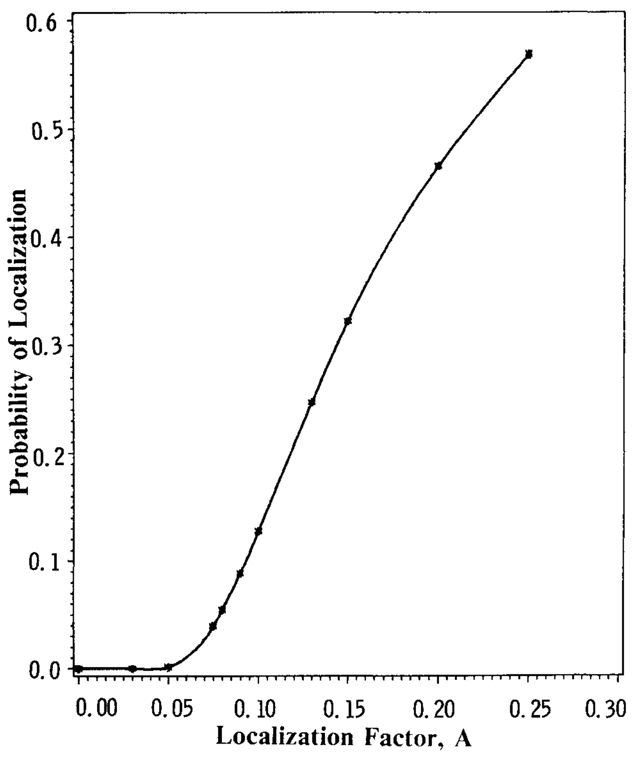

Fig. 9 Cumulative probability distribution function of the localization factor of the first mode of the two-span beam.

ation of possible structural failure places limits on the design variables. As before, there are two failure surfaces: one for positive $\overline{\Delta L}$ and another for negative $\overline{\Delta L}$, and the probability of failure of Eq. (12) is doubled such that

$$P_F = 2[1 - \Phi(\beta)] \tag{14}$$

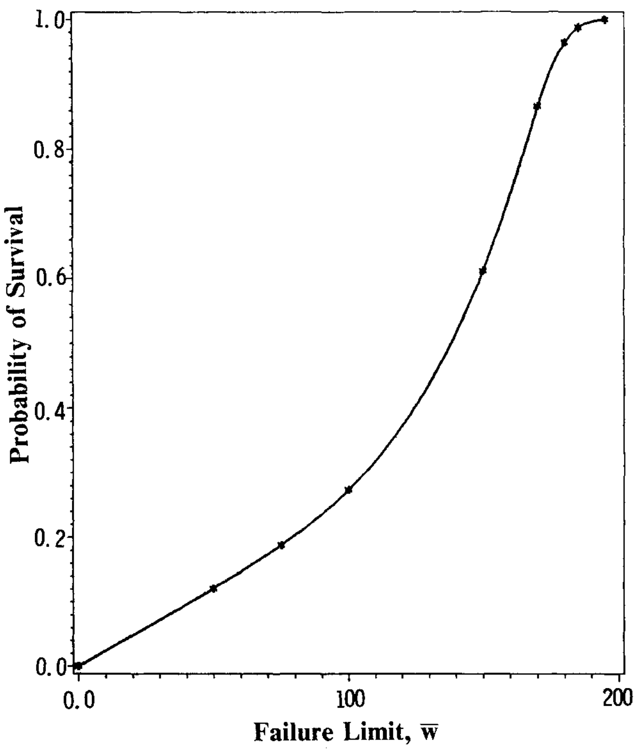

Fig. 10 Cumulative probability distribution function of the maximum forced vibration amplitude of the two-span beam.

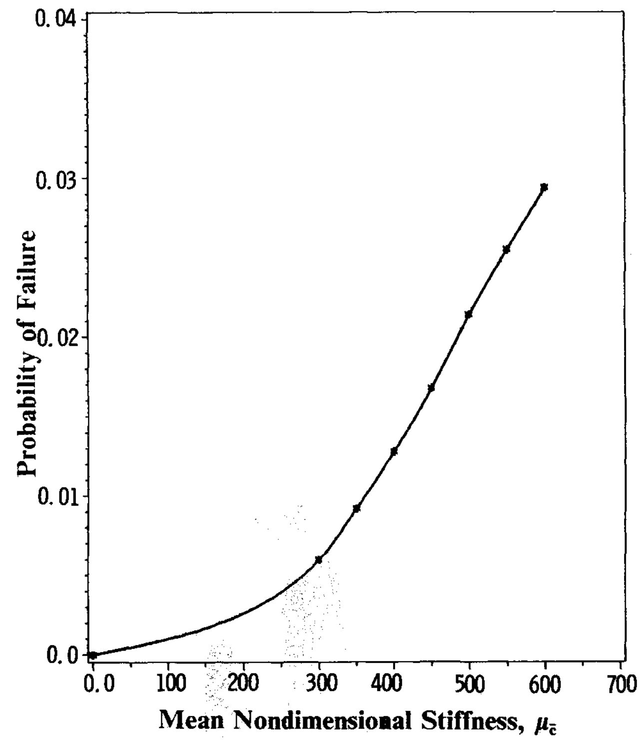

Fig. 11 Probability of failure of the two-span beam for varying mean nondimensional stiffness.

The corresponding probability of survival is

$$P_s = 1 - P_F \tag{15}$$

For example, if failure is defined to occur when the maximum nondimensional displacement exceeds $\bar{w} = 170$, then the performance function is $g = 170 - \bar{w}$. The second-moment method calculated a safety index β of 1.5017, which corresponds to a probability of failure of 0.1332. Therefore, there is an 87% chance that the in-phase forced response has a maximum response amplitude less than $\bar{w} = 170$.

In real life, high probabilities of survival (i.e., greater than 99%) are required for the design of a reliable structure. Figure 10 shows the probability of survival for varying failure limits, where the maximum allowable nondimensional amplitudes are set from 0 to 200. A study of the limiting cases showed that the probability of survival is equal to 1 (100%) when large response amplitudes are considered safe. However, the out-of-phase excitation of this mode resulted in an insensitivity to the degree of localization and a slight decrease in the response. Therefore, it is not necessary to evaluate the probability of failure since localization results in a lower chance of failure.

A further study is done to analyze the sensitivity of the probability of failure to changes in the mean values of the system parameters. The failure criterion is kept constant at $\bar{w} = 185$ in order to compare the probabilities of survival. In addition, the standard deviation of $\bar{c}$ remains 40.0, and the mean and standard deviations of $\overline{\Delta L}$ are 0.0 and 0.015, respectively. The results, which are plotted in Fig. 11 show that the probability of failure increases for increasing mean value of spring stiffness. For example, an increase of the mean stiffness from 400.0 to 500.0 results in an increase in the probability of failure from 0.0128 to 0.0214. The conclusion is that a system that is more weakly coupled in the mean is more likely to fail. This is true because the in-phase excitation of the localized first mode increases with stiffness and degree of mistuning. Similarly, changes in the standard deviation of the mistuning, $\sigma_{\overline{\Delta L}}$, result in changes in the probability of failure of the two-span beam (Fig. 12). When the standard deviation is increased from 0.010 to 0.015, the probability of failure increases from 0.0002 to 0.0128. As $\sigma_{\overline{\Delta L}}$ approaches zero, P_F likewise goes to zero. Such a diagram is useful in the design of a structure where the failure limit is known and tolerances

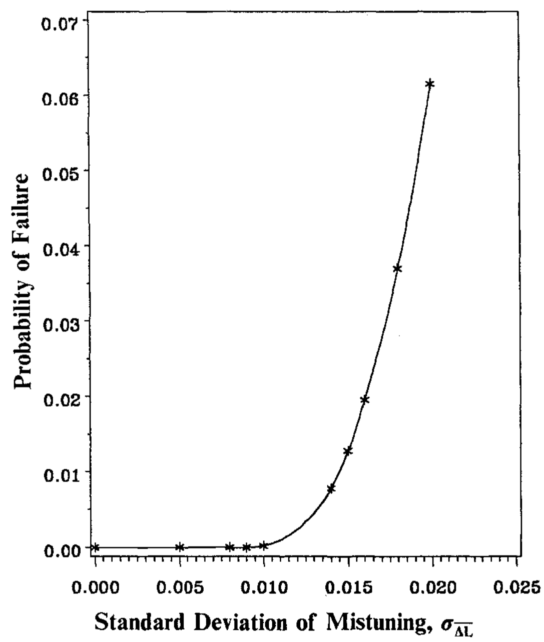

Fig. 12 Probability of failure of the two-span beam for varying standard deviation of nondimensional mistuning.

expressed in terms of the standard deviations of the system parameters must be considered.

IV. Discussion of Results

From the study of the two-span beam, the results show that the sensitivity of the forced response to the degree of localization depends on the particular combination of mode symmetry and forcing phases in the problem. The first mode of the two-span beam is antisymmetric with respect to the beam center support, while the second mode is symmetric. Furthermore, the in-phase excitation of an antisymmetric (first) mode causes the maximum response amplitude of the beam to increase dramatically when localization increases (Fig. 3). The out-of-phase excitation of a symmetric (second) mode of the beam also increases with localization (Fig. 6). The opposite conclusion is made for the out-of-phase excitation of an antisymmetric (first) mode and the in-phase excitation of a symmetric (second) mode of the beam (Figs. 4 and 5), in which case the maximum amplitude of the forced response is insensitive to the degree of localization in the modes. In fact, the response decreases slightly as the localization increases.

These results are parallel to the results of the forced response of the pair of nearly periodic coupled pendula where the first mode is symmetric and the second antisymmetric. For the coupled pendula the in-phase excitation of an antisymmetric (second) mode causes the maximum response of the masses to increase drastically with the degree of localization of that mode. Also, the out-of-phase excitation of a symmetric mode increases and is sensitive to the disorder in the system. However, other combinations of loading and mode symmetry result in the relative insensitivity of the forced response to localization: the in-phase excitation of a symmetric (first) mode and the out-of-phase excitation of an antisymmetric (second) mode are both relatively insensitive to the degree of localization. In the last two cases of insensitivity, the dynamic response can be evaluated by assuming a perfectly periodic structure since the analysis does not substantially change with the degree of localization. From the given observations it is concluded that the forced response of nearly periodic structures does not necessarily increase with the degree of localization. In some cases the magnitude of the maximum forced response amplitude may even decrease when the modes become localized.

A probabilistic analysis of the localization of modes of nearly periodic structures gives a quantitative measure of the chance of failure of mistuned structures due to excessive vibratory levels. This allows the designer who knows the approximate amount of disorder in a structure to predict the probability that the modes will be highly localized. When the system is likely to be localized, the probability of failure due to excessive vibration may be considerably higher than that in the case of a tuned system. Also, a study of the probability of failure for varying failure criteria indicates that survival is more likely when the failure limit is set at higher allowable maximum amplitudes. This general result is apparent; however, a graph showing the exact correlation between failure limits and survival probabilities is an important aid to the designer of nearly periodic structures.

V. Concluding Remarks

We summarize the major conclusions from this study as follows:

1) The response due to the in-phase excitation of an antisymmetric mode and the out-of-phase excitation of a symmetric mode increases drastically with an increase in localization of the mode.

2) The in-phase excitation of a symmetric mode and the out-of-phase excitation of an antisymmetric mode result in a forced response that is relatively insensitive to the localization of the mode and may even decrease.

3) For the case given in the preceding conclusion, the dynamic response can be evaluated assuming a perfectly ordered structure.

4) The first two observations lead to the conclusion that an increase in the localization of a mode does not necessarily mean that the forced response due to the excitation of that mode will also increase.

5) The second-moment method is appropriate for probabilistic analysis of free and forced vibration of nearly periodic structures. The advantage of the method is that it can be applied to study the vibration of real-life dynamic systems without requiring excessive computational effort, as do fully probabilistic methods. On the other hand, since the second-moment methods are far more accurate than first-order second-moment methods, they are considered by the authors to be the best compromise in terms of accuracy and computational efficiency. The second-moment method results for the structural systems in this paper were found to be in good agreement with the results of Monte Carlo simulation. It is hypothesized that the given conclusions can be generalized for the forced response of other more complex nearly periodic structures such as bladed disk assemblies.

In the present research, only the first two modes of the two-span beam are analyzed. From a study of these two modes, hypotheses are made concerning the dependence of the forced response on the symmetry of the mode excited in conjunction with the phases of the applied forces. Therefore, an area of additional work could be the study of the excitation of higher symmetric and antisymmetric modes. In addition, the probabilistic method could be applied to the analysis of more realistic nearly periodic structures such as bladed disk assemblies and large space structures. In the case of large space structures, the effect of localization in the forced vibration response should be an important consideration in active vibration control.

Appendix: Rayleigh-Ritz Method in Forced Response

Given the displacement of the beam in Eq. (2) and the tuned modes in Eq. (3), the Lagrangian of the two-span beam system can be written as

$$L = U - W_{nc} - T + \beta_1 f_1 + \beta_2 f_2 \tag{A1}$$

where β_1 and β_2 are the Lagrange multipliers. The virtual work of the nonconservative forces ∂W_{nc} is given by

$$\partial W_{nc} = F_1 \partial w(L/4, t) + F_2\, \partial w(3L/4, t) \tag{A2}$$

and $w(L/4, t)$ and $w(3L/4, t)$ are the displacements of the beam at the position of the applied forces. The terms U and T are the strain energy and kinetic energy, respectively, and they can be given by the following two expressions:

$$U = \frac{1}{2}\sum_{i=1}^{NM} \omega_i^2 a_i^2 + \frac{1}{2} c[w'(x_c)]^2 \tag{A3}$$

$$T = \frac{1}{2}\sum_{i=1}^{NM} \dot{a}_i^2 \tag{A4}$$

The constraints f_1 and f_2 are

$$f_1 = \sum_{i=1}^{NM} a_i(t)\phi_i(x_c) = 0 \tag{A5}$$

and

$$f_2 = w'(x_c) - \sum_{i=1}^{NM} a_i(t)\phi_i'(x_c) = 0 \tag{A6}$$

These constraints specify that the displacement of the beam at the position of the central support, x_c, is zero, and the slope of the beam is continuous at the central support. Then Hamilton's principle applied to Eq. (A1) gives the following equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{a}_i}\right) - \frac{\partial T}{\partial a_i} + \frac{\partial U}{\partial a_i} - \beta_1 \frac{\partial f_1}{\partial a_i} - \beta_2 \frac{\partial f_2}{\partial a_i} = Q_1 + Q_2 \qquad i = 1, \ldots, NM \qquad \text{(A7)}$$

where Q_1 and Q_2 are the generalized forces due to the applied loads. Equation (A7) can be evaluated in terms of the unknowns a_i, β_1, β_2, and $w'(x_c)$, which are, respectively, the generalized coordinates, Lagrange multipliers, and slope of the beam at the central support. These are Eqs. (A5), (A6) and

$$\ddot{a}_1 + \omega_i^2 a_i^2 - \beta_1 \phi_i(x_c) + \beta_2 \phi_i'(x_c) - F_1 \phi_i(L/4) - F_2 \phi_i(3L/4) = 0 \qquad i = 1, \ldots, NM \qquad \text{(A8)}$$

$$\beta_2 = c w'(x_c) \qquad \text{(A9)}$$

Simple harmonic motion is assumed with frequency Ω such that $a_i = \bar{a}_i e^{j\Omega t}$, $i = 1, \ldots, NM$, $\beta_k = \bar{\beta}_k e^{j\Omega t}$, and $k = 1, 2$ where $j = \sqrt{-1}$. The solution of the equations of motion gives the following expression for the amplitude of the generalized coordinate $\bar{a}_i$:

$$\bar{a}_i = \frac{1}{\omega_i^2 - \Omega^2} [\bar{\beta}_1 \phi_i(x_c) - \bar{\beta}_2 \phi_i'(x_c) + F e^{-\alpha} \phi_i(L/4) + F \phi_i(3L/4)] \qquad \text{(A10)}$$

Substituting Eq. (A10) into the constants f_1 and f_2 gives a system of two linear equations with unknowns of Lagrange multipliers $\bar{\beta}_1$ and $\bar{\beta}_2$. The value of $\bar{\beta}_1$ and $\bar{\beta}_2$ are used to evaluate the a_i and beam displacement. This displacement is nondimensionalized and evaluated along the dimensionless length of the beam $\bar{x} = x/L$ at time $t = 0$ such that

$$\bar{w}(\bar{x}, 0) = \sum_{i=1}^{NM} \frac{1}{\bar{\omega}_i^2 - \bar{\Omega}^2} [(\bar{\beta}_1/F) \sin(i\pi \bar{x}_c) - (\bar{\beta}_2 i\pi/FL) \cos(i\pi \bar{x}_c) + e^{-\alpha} \sin(i\pi/4) + \sin(3i\pi/4)] \sin(i\pi \bar{x}) \qquad \text{(A11)}$$

where the frequencies are nondimensionalized as given in Sec. II and

$$\bar{w}(\bar{x}, 0) = \frac{w(\bar{x}, 0)}{2L^2 F/EI} \qquad \text{(A12)}$$

Acknowledgments

The authors would like to give thanks for the support and helpful comments of Drs. J. A. Schetz, R. T. Haftka, R. K. Kapania, and E. R. Johnson of the Department of Aerospace and Ocean Engineering as well as Dr. M. P. Singh of the Engineering Science and Mechanics Department of Virginia Polytechnic Insitute and State University. This work was supported by the Department of Aerospace and Ocean Engineering whose assistance is deeply appreciated.

References

[1]Whitehead, D. S., "Effect of Mistuning on the Vibration of Turbomachine Blades Induced by Wakes," *Journal of Mechanical Engineering Science*, Vol. 8, No. 1, 1966, pp. 15–21.

[2]Srinivasan, A. V., "Vibrations of Bladed-Disk Assemblies—A Selected Survey," *Journal of Vibration, Acoustics, Stress and Reliability in Design*, Vol. 106, April 1984, pp. 165–167.

[3]El-Bayoumy, L. E., and Srinivasan, A. V., "Influence of Mistuning on Rotor-Blade Vibrations," *AIAA Journal*, Vol. 13, April 1975, pp. 460–464.

[4]Pierre, C., Tang, D. M., and Dowell, E. H., "Localized Vibrations of Disordered Multi-Span Beams: Theory and Experiment," AIAA Paper 86–0934, May 1986, pp. 445–455.

[5]Hodges, C. H., "Confinement of Vibration By Structural Irregularity," *Journal of Sound and Vibration*, Vol. 82, No. 3, 1982, pp. 411–424.

[6]Bendiksen, O. O., "Mode Localization Phenomena in Large Space Structures," AIAA Paper 86-0903, May 1986, pp. 325–335.

[7]Pierre, C., "Localized Free and Forced Vibrations of Nearly Periodic Disordered Structures," AIAA Paper 87–0774, April 1987, pp. 186–197.

[8]Valero, N. A., and Bendiksen, O. O., "Vibration Characteristics of Mistuned Shrouded Blade Assemblies," *Journal of Engineering for Gas Turbines and Power*, Vol. 108, April 1986, pp. 293–299.

[9]Sinha, A., "Calculating the Statistics of Forced Response of a Mistuned Bladed Disk Assembly," *AIAA Journal*, Vol. 24, Nov. 1986, pp. 1797–1801.

[10]Huang, W., "Vibration of Some Structures with Periodic Random Parameters," *AIAA Journal*, Vol. 20, July 1982, pp. 1001–1008.

[11]Cornwell, P. J. and Bendiksen, O. O., "Localization of Vibration in Large Space Structures," AIAA Paper 87–0949, April 1987, pp. 925–935.

[12]Pierre, C. and Dowell, E. H., "Localization of Vibrations by Structural Irregularity," *Journal of Sound and Vibration*, Vol. 114, No. 3, 1987, pp. 548–564.

[13]Ewins, D. J. and Han, Z. S., "Resonant Vibration Levels of a Mistuned Bladed Disk," *Journal of Vibration, Acoustics, Stress and Reliability in Design*, Vol. 106, April 1984, pp. 211–217.

[14]Macbain, J. C. and Whaley, P. W., "Maximum Resonant Response of Mistuned Bladed Disks," *Journal of Vibration, Acoustics, Stress and Reliability in Design*, Vol. 106, April 1984, pp. 218–223.

[15]Hoyniak, D and Fleeter, S., "Aerodynamic Detuning Analysis of an Unstalled Supersonic Turbofan Cascade," *ASME Journal of Engineering for Gas Turbines and Power*, Vol. 108, Jan. 1986, pp. 60–67.

[16]Kielb, R. E. and Kaza, K. R. V., "Aeroelastic Characteristics of a Cascade of Mistuned Blades in Subsonic and Supersonic Flows," ASME Paper 81-DET-122, Sept. 1981.

[17]Sogliero, G. and Srinivasan, A. V., "Fatigue Life Estimates of Mistuned Blades Via a Stochastic Approach," *AIAA Journal*, March 1980, Vol. 18, pp. 318–323.

[18]Ang, A. S. and Tang, W. H., *Probability Concepts in Engineering Planning and Design*, Vol. II, John Wiley and Sons, New York, 1984, pp. 340–361.

[19]Cornell, C. A., "Structural Safety Specifications Based on Second-Moment Reliability," Symposium of the International Association of Bridge and Structural Engineering, London, 1969.

[20]Ang, A. S. and Cornell, C. A., "Reliability Bases of Structural Safety and Design," *Journal of Structural Division*, Vol. 100, No. ST9, Sept. 1974, pp. 1755–1769.

[21]Hasofer, A. M. and Lind, N., "An Exact and Invariant First-Order Reliability Format," *Journal of Engineering Mechanics*, Vol. 100, No. EM1, Feb. 1974, pp. 111–121.

[22]Shinozuka, M., "Basic Analysis of Structural Safety," *Journal of Structural Division*, Vol. 109, No. ST3, March 1983, pp. 721–740.

[23]Rackwitz, R., "Practical Probabilistic Approach to Design," *Bulletin 112*, Comite European du Beton, Paris, France, 1976.